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LINEAR ANALYSIS OF QUADRATURE DOMAINS. II

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ABSTRACT

The natural correspondence between bounded planar quadrature domains, in the terminology of Aharonov–Shapiro, and certain square matrices with a distinguished cyclic vector is further exploited. Two different cubature formulas on quadrature domains, that is the computation of the integral of a real polynomial, are presented. The minimal defining polynomial of a quadrature domain is decomposed uniquely into a linear combination of moduli squares of complex polynomials. The geometry of a canonical rational embedding of a quadrature domain into the projective complement of a real affine ball is also investigated. Explicit computations on order-two quadrature domains illustrate the main results.

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Introduction

This note is a continuation of [P1] and it is devoted to some constructive aspects of the relation between quadrature domains and their linear data. We assume that the defining polynomial of a bounded quadrature domain is given and we try to find explicit formulas for the real moments of the domain and other naturally associated objects. Our approach is based on the observation that a quadrature domain is the level set of the norm of the resolvent of a square matrix, localized at a specific cyclic vector.

The first set of formulas we propose starts from the equation of a quadrature domain, it passes through an inversion of a Hankel matrix (formula (11) in text) and requires a logarithm of formal series (formula (12)) in order to compute all the moments of the domain.

The second method of computing the same moments starts again from the equation of the quadrature domain, then it identifies from this equation a square matrix with a cyclic vector, called in the sequel the linear data of the domain, and finally exploits the Helton–Howe trace formula for seminormal operators in order to evaluate the moments. In particular, this method gives a non-commutative cubature formula on quadrature domains (formula (15) in text) which is exact on all n-polyharmonic polynomials, for n specified. An error formula for this cubature is then obtained.

The rest of the paper deals with some specific properties of the resolvent of the linear data of a quadrature domain. The minimal polynomial which defines a quadrature domain of order d is canonically decomposed into the modulus square of the minimal polynomial of the associated matrix minus exactly d moduli squares of complex polynomials, of exact degrees $d-1, d-2, \ldots, 1, 0$. Thus a natural set of parameters of a quadrature domain is exhibited.

A canonical rational embedding of the quadrature domain Ω of order d in the exterior of the unit ball of $\mathbf{C}^d$ is obtained. Then we prove that the multivalued Schwarz reflection in the boundary of Ω maps the exterior of Ω into $\overline{\Omega}$, and in this transformation the boundary covers the boundary exactly once via the identity map. A uniqueness result for this embedding of a quadrature domain in the exterior of a multidimensional ball is also established. As a consequence certain rational maps from $\mathbf{C}$ into $\mathbf{C}^d$ which commute with the reflections in the unit balls of the two affine spaces are classified.

A few simple examples of the interplay between planar domains and pairs of matrices with a cyclic vector end the paper.

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1. Preliminaries

We recall from [P1] a few formulas which relate a quadrature domain to a matrix with a cyclic vector. Although these formulas have been motivated by the study of the L-problem of moments in the real plane, we do not make any precise reference to this relationship; see for details [P2].

Let Ω be a bounded planar domain and let dA stand for the area measure in $\mathbf{C}$. The coordinate in the complex plane $\mathbf{C}$ will be denoted by z. The domain Ω is called, following the terminology of Aharonov and Shapiro [AS], a **quadrature domain** if there exists a distribution u with finite support in Ω such that

$$\int_\Omega f dA = u(f),$$

for every integrable analytic function f in Ω. Quadrature domains tend to be very rigid; they are remarkable in many respects as it is amply illustrated by the recent monograph [Sh].

The **order** of the quadrature domain Ω is the cardinality of the support of u, counting multiplicities. To be more specific, there are points $\lambda_j \in \Omega$ and constants $\gamma_{jk}, 0 \le k \le m(j)-1, 1 \le j \le m$, with the property that

$$u(f) = \sum_{j=1}^{m} \sum_{k=0}^{m(j)-1} \gamma_{jk} f^{(k)}(\lambda_j),$$

where u and f are as above. To make the above decomposition optimal, we assume that $\gamma_{j,m(j)-1} \neq 0$ for all $j, 1 \le j \le m$. The order $d = d(\Omega)$ of Ω is then by definition

$$d = \sum_{j=1}^{m} m(j).$$

The quadrature domains of order one are precisely the disks; see [Sh]. In general the equation of the boundary of a quadrature domain Ω of order d is given, up to a finite set, by the equation $Q(z, \overline{z}) = 0$, where

$$Q(z,\overline{z}) = \sum_{k,l=0}^{d} \alpha_{kl} z^k \overline{z}^l \tag{1}$$

is a monic self-adjoint irreducible polynomial; by self-adjoint we mean $\alpha_{kl} = \overline{\alpha_{lk}}$ and by monic we mean $\alpha_{dd} = 1$. For details we refer to [AS], [G1].

A quadrature domain Ω is characterized by the existence of a meromorphic function $S(z)$ in Ω, continuous on $\overline{\Omega} \setminus \{\lambda_1, \ldots, \lambda_m\}$, with the property

$$S(z) = \overline{z} \quad \text{for } z \in \partial\Omega. \tag{2}$$

The function S is called **the Schwarz function** of Ω; see [D] and [Sh]. The poles of $S(z)$ coincide, including the multiplicities, with the nodes $\lambda_j, 1 \leq j \leq m$, of the quadrature identity. Let us define the polynomial:

$$P(z) = \prod_{j=1}^{m} (z - \lambda_j)^{m(j)},$$

so that $P(z)S(z)$ is a holomorphic function in Ω.

The following facts were established in [G1].

THEOREM 1.1 ([G1], Section 6): *Let Ω be a quadrature domain. Then, with the above notation, we have*

$$P(z) = z^d + \sum_{j=0}^{d-1} \alpha_{jd} z^j, \tag{3}$$

and

$$\frac{1}{\pi} \sum_{k=1}^{m} \sum_{l=0}^{m(k)-1} \frac{l! \gamma_{kl}}{(z - \lambda_k)^{l+1}} = \alpha_{d,d-1} - \frac{\sum_{j=0}^{d} \alpha_{j,d-1} z^j}{P(z)} = S(z) + A(z), \tag{4}$$

where $A(z)$ is an analytic function in Ω.

An explanation of these formulas will become available later in this and the next section. Roughly speaking, Theorem 1.1 above asserts that the first two lines in the matrix of coefficients α_{kl} of the defining polynomial $Q(z, \overline{z})$ determine the quadrature data λ_j, γ_{jk}, as well as the polar part of the Schwarz function $S(z)$.

Actually, there is more structure in the defining polynomial Q. Namely, there exists a linear transformation U: $\mathbf{C}^d \longrightarrow \mathbf{C}^d$ with a cyclic vector $\xi \in \mathbf{C}^d$ for U^* and with $P(z)$ as minimal and (up to a sign) characteristic polynomial, such that:

$$\frac{Q(z, \overline{z})}{|P(z)|^2} = 1 - \|(U^* - \overline{z})^{-1} \xi\|^2, \tag{5}$$

where the equality is understood in the sense of rational functions; see for details [P1]. It is clear from the above discussion that both the polynomial $Q(z, \overline{z})$ or the pair (U, ξ) form complete invariants for the quadrature domain Ω. Since the leading coefficient of $Q(z, \bar{z})$ is positive, Ω is in fact given, up to a finite set, by

$$\Omega = \{z \in \mathbf{C}; Q(z, \bar{z}) < 0\} = \{z \in \mathbf{C}; \|(U^* - \bar{z})^{-1}\xi\| > 1\}. \tag{6}$$

In the other direction, $Q(z, \bar{z})$ is uniquely determined by Ω while (U, ξ) is determined only up to unitary equivalence.

Similarly to Theorem 1.1 we have the following result.

THEOREM 1.2 ([P1] Section 3): *Let Ω be a quadrature domain. With the above notation we have*

$$u(f) = \pi\langle f(U)\xi, \xi\rangle, \tag{7}$$

for every analytic function f in Ω. Moreover,

$$S(z) = -\langle (U - z)^{-1}\xi, \xi\rangle - A(z), \tag{8}$$

where $A(z)$ is an analytic function in Ω (the same as in Theorem 1.1).

Notation: Throughout this paper we keep generically unchanged the notation introduced in this section. That is, Ω is a bounded quadrature domain of order d and the quadrature data are $\lambda_j, \gamma_{jk}, 1 \le j \le m, 0 \le k \le m(j) - 1$. The defining polynomial of Ω is $Q(z, \bar{z})$ with the coefficients $\alpha_{jk}, 0 \le j, k \le d, \alpha_{dd} = 1$. The Schwarz function is $S(z)$ with denominator $P(z)$, and the linear data of Ω are (U, ξ). In addition, we will consider the moments of the domain Ω

$$a_{mn} = \int_\Omega z^m \bar{z}^n dA(z),$$

and the scalar products

$$g_{mn} = \langle U^{*n}\xi, U^{*m}\xi\rangle,$$

where m, n are non-negative integers.

2. From the equation of a quadrature domain to its linear data

The aim of this section is to find explicit formulas for computing the integral of a polynomial in z and $\bar{z}$ on a quadrature domain Ω (against the area measure), knowing only the defining polynomial of the boundary of Ω.

Let us recall the basic exponential transformation which relates the finite matrix $(\alpha_{jk})_{j,k=0}{}^d$ to the infinite matrix $(a_{mn})_{m,n=0}{}^\infty$:

$$\frac{Q(z, \bar{z})}{|P(z)|^2} = \exp\left(\frac{-1}{\pi} \sum_{m,n=0}^{\infty} \frac{a_{mn}}{z^{m+1}\bar{z}^{n+1}}\right), \tag{9}$$

which is valid for large values of $|z|$; see for details [P2]. Thus, by taking a logarithm at the level of formal series, the moments a_{mn} can be determined from the defining polynomial $Q(z,\overline{z})$. In its turn, the pair of matrices (U,ξ) can be used in simplifying the above computation:

$$\frac{Q(z,\overline{z})}{|P(z)|^2} = 1 - \sum_{m,n=0}^{\infty} \frac{\langle U^{*n}\xi, U^{*m}\xi\rangle}{z^{m+1}\overline{z}^{n+1}} = 1 - \sum_{m,n=0}^{\infty} \frac{g_{mn}}{z^{m+1}\overline{z}^{n+1}}. \tag{10}$$

Therefore, a direct relation between the matrix of coefficients α_{kl} and the Gram matrix g_{mn} becomes possible. To simplify the following computation we put $\alpha_k = \alpha_{kd}, 0 \le k \le d$, so that $P(z) = \sum_{k=0}^{d} \alpha_k z^k$. Note that $\alpha_d = 1$ by a convention we have adopted in the previous section.

We begin with a series of elementary computations:

$$\overline{P(z)}(U^* - \overline{z})^{-1}\xi = (\overline{P(z)} - P(U)^*)(U^* - \overline{z})^{-1}\xi = -\sum_{k=1}^{d} \overline{\alpha}_k (\sum_{s=0}^{k-1} U^{*k-s-1}\xi\overline{z}^s).$$

Later we will return to a second possible form of the same polynomial (see formula (19) below).

Accordingly we obtain

$$|P(z)|^2 \|(U^* - \overline{z})^{-1}\xi\|^2 = \sum_{k,l=1}^{d} \sum_{s=0}^{k-1} \sum_{t=0}^{l-1} \overline{\alpha_k}\alpha_l \langle U^{*k-s-1}\xi, U^{*l-t-1}\xi\rangle z^t \overline{z}^s.$$

If we fix s,t in the last formula and perform the other summations, we obtain by (5) the coefficient of $z^t\overline{z}^s$ in $|P(z)|^2 - Q(z,\overline{z})$.

Let $G = (g_{jk})_{j,k=0}^{d-1}$ and let us also introduce the Hankel matrix

$$H(\alpha) = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \dots & 1 \\ \alpha_2 & \alpha_3 & \alpha_4 & \dots & 0 \\ \alpha_3 & \alpha_4 & \alpha_5 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_{d-1} & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

and the matrix $A(\alpha)$ of coefficients of the polynomial $|P(z)|^2 - Q(z,\overline{z})$:

$$A(\alpha)_{jk} = \alpha_j\overline{\alpha_k} - \alpha_{jk} \quad (0 \le j,k \le d-1).$$

Then we obtain

$$A(\alpha)_{jk} = \sum_{r,s} \alpha_r \overline{\alpha}_s g_{r-j-1,s-k-1} = \sum_{p,q} \alpha_{j+p+1} g_{pq} \overline{\alpha}_{k+q+1}.$$

The above computation can be summarized in the following result.

PROPOSITION 2.1: *The Gram matrix G of the linear data (U,ξ) of a quadrature domain can be obtained from the coefficients $(\alpha_{jk})_{j,k=0}{}^{d}$ of the defining polynomial by the formula*

$$H(\alpha)GH(\alpha)^* = A(\alpha). \tag{11}$$

Clearly Proposition 2.1 implies that the matrix $A(\alpha)$ is positive definite, since G is. Later on (Theorem 4.3) we shall see, conversely, that a matrix of the form $A(\alpha)$ being positive definite is not only necessary, but also sufficient, for the existence of linear data (U,ξ) related to $A(\alpha)$ as in (10) or (11).

Finally, let us write the announced formula for the moments of a quadrature domain:

$$\sum_{m,n=0}^{\infty} \frac{a_{mn}}{z^{m+1}\bar{z}^{n+1}} = -\pi \log\left(1 - \sum_{m,n=0}^{\infty} \frac{g_{mn}}{z^{m+1}\bar{z}^{n+1}}\right). \tag{12}$$

We remark that the above transformation, from the matrix (g_{mn}) to the matrix of moments (a_{mn}), is triangular in the sense that a_{mn} depends only on g_{kl} where $0 \le k \le m$ and $0 \le l \le n$. A couple of examples of low order quadrature domains which illustrate the preceding formulas are included in the last section of the paper.

Our next aim is to factor the Gram matrix G into the linear data (U,ξ) and then to use them in another formula for the moments of the quadrature domain, this time the computations being carried only at the level of linear algebra (and avoiding non-linear operations such as the above logarithm).

3. A non-commutative cubature formula

In this section we exploit the Helton–Howe trace formula in the construction of a cubature formula on quadrature domains. Traditionally, cubature formulas in one or several variables arise from the evaluation of functions at the zeroes of some families of orthogonal polynomials; see [ST], [Xu]. Below we approximate the integral of a (real analytic) function on a quadrature domain Ω by its values on the matrix U and some bigger matrices constructed recurrently from U. An error formula is obtained, similar to the errors in the well studied one dimensional theory; see [ST] Chapter IV.

Let Ω be a quadrature domain of order d with the linear data (U,ξ) on the Hilbert space K of dimension d. Let T be the unique irreducible hyponormal operator, acting on the Hilbert space $H, K \subset H$, such that $[T^*,T] = \xi \otimes \xi$

and with principal function equal to the characteristic function of Ω; see for details [P1], [P2]. We recall from [P1] that K is the linear span of the vectors $\{T^{*n}\xi; n \geq 0\}$ and that $U^* = T^*|K$.

For a polynomial $p \in \mathbf{C}[z, \overline{z}]$,

$$p(z,\overline{z}) = \sum_{\alpha+\beta\leq n} c_{\alpha\beta} z^{\alpha}\overline{z}^{\beta},$$

we introduce the symmetrized operator valued functional calculus:

$$p^{\sharp}(T,T^*) = \sum_{\alpha+\beta\leq n} \frac{c_{\alpha\beta}}{\beta+1} \sum_{\gamma=0}^{\beta} T^{*\gamma}T^{\alpha}T^{*\beta-\gamma}. \tag{13}$$

LEMMA 3.1: *With the above notation we have*

$$\int_{\Omega} p dA = \pi \langle p^{\sharp}(T,T^*)\xi, \xi\rangle. \tag{14}$$

Proof: Indeed, for a monomial $z^n\overline{z}^m$, the Helton–Howe trace formula (see for details [P2]) yields

$$\pi^{-1}\int_{\Omega} z^n\overline{z}^m dA = \frac{1}{m+1}\operatorname{Tr}[T^{*m+1}T^n, T]$$

$$= \frac{1}{m+1}\operatorname{Tr}\sum_{k=0}^{m} T^{*k}[T^*,T]T^{*m-k}T^n = \frac{1}{m+1}\sum_{k=0}^{m}\langle T^{*m-k}T^nT^{*k}\xi,\xi\rangle. \qquad \blacksquare$$

The previous formula becomes effective as soon as we recall the block structure of the operator T. To be more precise, let us define recurrently

$$U_0 = U, \quad {A_0}^2 = \xi\otimes\xi - [{U_0}^*, U_0],$$

and for $k \geq 0$,

$$U_{k+1} = {A_k}^{-1}U_kA_k, \quad {A_{k+1}}^2 = {A_k}^2 - [{U_{k+1}}^*, U_{k+1}].$$

We know from [P1], Theorem 4.2 that, for all $k \geq 0$, A_k are positive matrices on the space K. Then the operator T is unitarily equivalent to an infinite block matrix with U_k on the diagonal, A_k under the diagonal and zero elsewhere.

For a fixed positive integer n we denote by T_n the $(n+1)\times(n+1)$-block truncation of T. More specifically

$$T_n = \begin{pmatrix} U_0 & 0 & 0 & \dots & 0 & 0 \\ A_0 & U_1 & 0 & \dots & 0 & 0 \\ 0 & A_1 & U_2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & U_{n-1} & 0 \\ 0 & 0 & 0 & \dots & A_{n-1} & U_n \end{pmatrix}.$$

For a polynomial $p(z,\overline{z})$ we denote by $\deg_z(p), \deg_{\overline{z}}(p)$ the corresponding degrees in z and $\overline{z}$.

THEOREM 3.2: *Let Ω be a quadrature domain with associated hyponormal operator T and let $p \in \mathbf{C}[z,\overline{z}]$. Then*

$$\int_\Omega p\,dA = \pi\langle p^\sharp(T_n, {T_n}^*)\xi, \xi\rangle \tag{15}$$

whenever $n \geq \min(\deg_z(p), \deg_{\overline{z}}(p))$.

Proof: Let P_n denote the orthogonal projection of the Hilbert space H onto the finite dimensional subspace $K_n = K + TK + \cdots + T^nK$. Then we find the identities $T_n = P_nTP_n$ and $T^*P_n = P_nT^*P_n$ from the block structure of the matrix T. Moreover, $T^kx = (P_nTP_n)^kx = {T_n}^kx$ for every $x \in K$ and $k \leq n$.

Let $p(z,\overline{z})$ be a polynomial satisfying $\deg_z(p) \leq n$. For a typical monomial in $p^\sharp(T,T^*)$ we have

$$\langle T^{*\gamma}T^\alpha T^{*\beta-\gamma}\xi,\xi\rangle = \langle T^{*\gamma}P_nT^\alpha P_0T^{*\beta-\gamma}\xi,\xi\rangle = \langle {T_n}^{*\gamma}{T_n}^\alpha {T_n}^{*\beta-\gamma}\xi,\xi\rangle,$$

because $\alpha \leq n$.

Similarly, assume instead that $\deg_{\overline{z}}(p) \leq n$. Then, in the above notation $\gamma \leq \beta \leq n$, whence

$$\langle T^{*\gamma}T^\alpha T^{*\beta-\gamma}\xi,\xi\rangle = \langle P_0T^{*\gamma}P_nT^\alpha P_0T^{*\beta-\gamma}\xi,\xi\rangle = \langle {T_n}^{*\gamma}{T_n}^\alpha {T_n}^{*\beta-\gamma}\xi,\xi\rangle.$$

This completes the proof of Theorem 3.2. ∎

Let us remark that for analytic polynomials $p(z)$, formula (15) reduces to the quadrature identity (7). In the spirit of some recent advances in multivariable cubature formulas (cf. [Xu]), relation (15) holds in particular for $\deg(p) \leq 2n+1$.

For an arbitrary polynomial p, the error in formula (15) depends only on the monomials in p of the form $z^\alpha\overline{z}^\beta$ with both α and β strictly larger than n, hence only on $\Delta^{n+1}p$, where Δ is the Laplace operator. Actually we can make this statement more precise.

For a disk $D(0,\rho)$ centered at zero, of radius ρ and a polynomial $p(z) = \sum_{\alpha+\beta\leq N} c_{\alpha\beta}z^\alpha\overline{z}^\beta$ we introduce the norm

$$\|p\|_\rho = \sum_{\alpha+\beta\leq N} |c_{\alpha\beta}|\rho^{\alpha+\beta}.$$

In virtue of Cauchy inequalities for functions of two variables, for every positive ϵ, the preceding norm can be estimated from above by the uniform norm of $p(z,w)$ for $|z|,|w| \leq \rho+\epsilon$. However, we do not make use of this estimate below.

PROPOSITION 3.3: *Let Ω be a quadrature domain contained in the disk $D(0,\rho)$ and let $p(z,\overline{z})$ be an arbitrary polynomial.*

Then for every positive integer n we have

$$(16)\qquad |\pi^{-1}\int_\Omega p\,dA - \langle p^\sharp(T_n,T_n{}^*)\xi,\xi\rangle| \le \frac{\text{Area}(\Omega)}{\pi}\frac{(\rho/2)^{2n+2}}{(n+1)!^2}\|\Delta^{n+1}p\|_\rho.$$

Proof: Since the domain Ω is contained in the disk $D(0,\rho)$, the spectral radius of the operator T is less than or equal ρ. But for hyponormal operators this implies $\|T\| \le \rho$; see [P2] and the references cited there.

According to Lemma 3.1 we have to estimate

$$|\langle(p^\sharp(T,T^*) - p^\sharp(T_n,T_n{}^*))\xi,\xi\rangle|.$$

For a typical monomial in this expression we obtain

$$|\langle(T^{*\gamma}T^\alpha T^{*\beta-\gamma} - T_n{}^{*\gamma}T_n{}^\alpha T_n{}^{*\beta-\gamma})\xi,\xi\rangle| =$$

$$|\langle T^{*\gamma}(I-P_n)T^\alpha T^{*\beta-\gamma}\xi,\xi\rangle| \le \|T\|^{\alpha+\beta}\|\xi\|^2 \le \frac{\text{Area}(\Omega)}{\pi}\rho^{\alpha+\beta}.$$

Let $p(z,\overline{z})$ be as above, with coefficients $c_{\alpha\beta}$. Then

$$\begin{aligned}\frac{\Delta^{n+1}p}{4^{n+1}} &= \partial^{n+1}\overline{\partial}^{n+1}p\\ &= \sum_{\alpha,\beta>n} c_{\alpha\beta}\alpha\beta(\alpha-1)(\beta-1)\cdots(\alpha-n)(\beta-n)z^{\alpha-n-1}\overline{z}^{\beta-n-1}.\end{aligned}$$

On the other hand,

$$|\langle(p^\sharp(T,T^*) - p^\sharp(T_n,T_n{}^*))\xi,\xi\rangle|$$

$$\begin{aligned}&= |\sum_{\alpha,\beta>n}\frac{c_{\alpha\beta}}{\beta+1}\sum_{\gamma=0}^{\beta}\langle T^{*\gamma}(I-P_n)T^\alpha T^{*\beta-\gamma}\xi,\xi\rangle| \le \|\xi\|^2\sum_{\alpha,\beta>n}|c_{\alpha\beta}|\rho^{\alpha+\beta}\\ &\le \frac{\|\xi\|^2\rho^{2n+2}}{4^{n+1}}\sum_{\alpha,\beta>n}\frac{4^{n+1}|c_{\alpha\beta}|\alpha\beta(\alpha-1)(\beta-1)\cdots(\alpha-n)(\beta-n)\rho^{\alpha+\beta-2n-2}}{(n+1)!(n+1)!}\\ &\le \frac{\text{Area}(\Omega)}{\pi}\frac{(\rho/2)^{2n+2}}{(n+1)!^2}\|\Delta^{n+1}p\|_\rho.\end{aligned}$$

This completes the proof of Proposition 3.3. ∎

Suppose that $f(z,\overline{w})$ is an analytic function of two complex variables which is convergent in the polydisk $D(0,\rho')\times D(0,\rho')$, where $\rho' > \rho$. Then, by restricting f to the diagonal, the norm $\|f\|_\rho$ is finite. This shows in particular that the functional calculus $f^\sharp(T_n, T^*{}_n)$ makes sense; and so does Proposition 3.3. Thus the quadrature formula (15) is exact for such functions which, in addition, are $(n+1)$-polyharmonic when restricted to the diagonal.

Unfortunately, for a general $(n+1)$-polyharmonic function $f(z,\overline{z})$ in a neighbourhood of $\overline{\Omega}$, or even in $D(0,\rho')$, $f^\sharp(T_n, T_n^*)$ does not necessarily make sense, because even though $f(z,\overline{w})$ extends analytically to some neighbourhood of the diagonal it may happen that it does not extend to all of $D(0,\rho') \times D(0,\rho')$. Cf. [V], [KS].

4. A rational embedding of linear data into projective space

The resolvent of the matrix associated to a quadrature domain of order d gives a canonical embedding into an affine, or projective, complex space of dimension d. This embedding will prove to be functorial with respect to the reflections in the boundaries of the original quadrature domain and respectively the unit sphere in $\mathbf{C}^d$.

The projective space of dimension n will be denoted by $\mathbf{P}_n(\mathbf{C})$ or simply $\mathbf{P}_n$, and projective (or homogeneous) coordinates in $\mathbf{P}_n$ are written like $(z_0 : z_1 : \ldots : z_n)$, or sometimes $(z_0 : z)$ where z denotes the vector

$$z = (z_1, \ldots, z_n).$$

We always consider $\mathbf{C}^n$ embedded in $\mathbf{P}_n$ by $(z_1,\ldots,z_n) \mapsto (1 : z_1 : \ldots : z_n)$ so that, in particular, $\mathbf{P}_n$ is provided with a specific origin $0 = (1:0:\ldots:0) \in \mathbf{P}_n$, and $\mathbf{P}_1$ also with a specific point of infinity, $\infty = (0:1) \in \mathbf{P}_1$.

In this section we treat a slightly more general situation than that encountered in the case of quadrature domains. Namely, let n be a positive integer, $n > 1$, let A be a linear transformation of $\mathbf{C}^n$ with spectrum denoted $\sigma(A)$, and let $\xi \in \mathbf{C}^n$ be a nonzero vector. Let us denote

$$R(z) = (A-z)^{-1}\xi, \qquad z \in \mathbf{C} \setminus \sigma(A), \tag{17}$$

the resolvent of A, localized at the vector ξ.

LEMMA 4.1: *The map R: $\mathbf{C} \setminus \sigma(A) \longrightarrow \mathbf{C}^n$ is one to one and its range is a smooth complex curve.*

Proof: Indeed, according to the resolvent equation we find

$$R(z) - R(w) = (z-w)(A-z)^{-1}(A-w)^{-1}\xi, \quad z,w \in \mathbf{C} \setminus \sigma(A).$$

Thus $R(z) - R(w) \neq 0$ for $z \neq w$. For the point at infinity we have $R(\infty) = 0 \neq R(z)$, for $z \in \mathbf{C} \setminus \sigma(A)$.

Moreover, the same resolvent equation shows that

$$R'(z) = (A - z)^{-1} R(z) \neq 0,$$

and similarly for the point at infinity we obtain

$$\frac{d}{dt} R(1/t) = -\lim_{t \to 0} [t^{-2}(A - t^{-1})^{-2} \xi] = -\xi \neq 0. \qquad \blacksquare$$

Next we pass to projective spaces and complete the above curve. Let d be the largest number such that ξ, $A\xi, \ldots$, $A^{d-1}\xi$ are linearly independent. Then there is a monic polynomial $P(z) = z^d + \alpha_{d-1} z^{d-1} + \cdots + \alpha_0$ satisfying

$$P(A)\xi = 0$$

and no such polynomial of lower degree. Set $\alpha_d = 1$ and define polynomials $T_k(z)$ $(0 \leq k \leq d)$ by

$$T_k(z) = z^k + \alpha_{d-1} z^{k-1} + \cdots + \alpha_{d-k+1} z + \alpha_{d-k}. \tag{18}$$

These are the polynomials appearing in the difference quotients $q(z,w)$ of $P(z)$:

$$\begin{aligned} q(z,w) = \frac{P(w) - P(z)}{w - z} &= \sum_{k=0}^{d} \alpha_k \frac{w^k - z^k}{w - z} \\ &= \sum_{k=0}^{d} \alpha_k \sum_{j=0}^{k-1} z^{k-j-1} w^j = \sum_{j=0}^{d-1} \Big(\sum_{k=j+1}^{d} \alpha_k z^{k-j-1} \Big) w^j \\ &= T_0(z) w^{d-1} + T_1(z) w^{d-2} + \cdots + T_{d-1}(z). \end{aligned}$$

Note that $T_0(z) = 1$, $T_d(z) = P(z)$.

Since

$$\begin{aligned} P(z)(A - z)^{-1}\xi &= -(A - z)^{-1}(P(A) - P(z))\xi = -q(z, A)\xi \\ &= -[T_0(z) A^{d-1} + T_1(z) A^{d-2} + \cdots + T_{d-1}(z)]\xi, \end{aligned} \tag{19}$$

$R(z)$ is given, using projective coordinates in $\mathbf{P}_n$, by

$$\begin{aligned} R(z) &= (P(z) : P(z)(A - z)^{-1}\xi) = (P(z) : -q(z, A)\xi) \\ &= (P(z) : -T_0(z) A^{d-1}\xi - T_1(z) A^{d-2}\xi - \cdots - T_{d-1}(z)\xi). \end{aligned} \tag{20}$$

Recall that ξ, $A\xi,\ldots,A^{d-1}\xi$ are linearly independent. Using them (with a minus sign) as a basis for the subspace spanned by the image of R we get R represented as

$$R(z) = (T_d(z) : T_{d-1}(z) : \ldots : T_0(z) : 0 : \ldots : 0).$$

Here each $T_k(z)$ has degree k exactly, hence a further triangular change of the homogeneous coordinates (giving a projective change of coordinates in $\mathbf{P}_n$ which preserves the embedding $\mathbf{C}^n \subset \mathbf{P}_n$) brings R onto the form

$$R(z) = (z^d : z^{d-1} : \ldots : 1 : 0 : \ldots : 0)$$

or, using homogeneuos coordinates also in $\mathbf{P}_1$,

$$R(z_0 : z_1) = (z_1^d : z_0 z_1^{d-1} : \ldots : z_0^d : 0 : \ldots : 0).$$

Thus, after a change of coordinates and considering R as a map into the projective space $\mathbf{P}_d$ spanned by its image, R simply becomes an instance of the classical Veronese embedding (see [GH, p. 178f]), and the image curve $R(\mathbf{P}_1)$ hence is projectively isomorphic to the rational normal curve of degree d (which by definition is the image of the Veronese embedding).

In particular it follows that R is a smooth rational embedding of degree d and that its image spans a projective space of dimension d. It is well-known that in fact every rational map of degree d whose image spans a space of dimension d is projectively isomorphic to the Veronese embedding of degree d. Thus, what we did above is essentially that we performed the transformation explicitly in our case. This could also have been done using methods from realization theory; see e.g. [BGR].

By the above the first half of the following theorem is proved.

THEOREM 4.2: *Let A be a linear transformation of $\mathbf{C}^n$ and let ξ be a non-zero vector of $\mathbf{C}^n$. Then the map $R(z) = (A-z)^{-1}\xi$ extends to a rational embedding:*

$$R\colon \mathbf{P}_1 \longrightarrow \mathbf{P}_n.$$

The range of R is contained in the projective completion of $E = \bigvee_{k=0}^{\infty} A^k\xi$ and the values $R(z)$ span E as a linear space. Moreover, the degree of R equals the dimension of E.

Conversely, any rational map $R\colon \mathbf{P}_1 \to \mathbf{P}_n$ such that the degree of R equals the dimension of the subspace spanned by its image and such that $R(\infty) = 0$ is given by (the projective completion of) $R(z) = (A-z)^{-1}\xi$ for some linear transformation A of $\mathbf{C}^n$ and some non-zero vector ξ in $\mathbf{C}^n$.

Remark on terminology: By a rational map R: $\mathbf{P}_1 \longrightarrow \mathbf{P}_n$ in general we mean a map which in homogenous coordinates can be written

$$R(z_0 : z_1) = (P_0(z_1/z_0) : P_1(z_1/z_0) : \ldots : P_n(z_1/z_0))$$

where $P_0, ..., P_n$ are polynomials, not all identically zero. Assuming (as we can) that the P_j have no common factor, the degree of the map is the highest occurring degree of the P_j. Then the polynomials have no common zero, hence R is well-defined at every point. By rational embedding we mean a rational map which is one-to-one and has nonvanishing derivative.

Proof of Theorem 4.2: It only remains to prove the converse part. For this it is possible to invoke classical results on the Veronese embedding, but we prefer to argue directly.

We may consider R as a map from $\mathbf{P}_1$ to the projective subspace spanned by its image, thus we may simply assume that R: $\mathbf{P}_1 \to \mathbf{P}_d$ has degree d and that $R(\mathbf{P}_1)$ spans $\mathbf{P}_d$.

Since $R(\infty) = 0$, R has the form

$$R(z) = (P_0(z) : P_1(z) : \ldots : P_d(z))$$

with $\deg P_0 = d$, $\deg P_j \leq d-1$ for $1 \leq j \leq d$, and we may assume that $P_0(z)$ is monic. Write

$$P_0(z) = z^d + \alpha_{d-1}z^{d-1} + \cdots + \alpha_d$$

and define $T_k(z)$ in terms of the α_k as in (18). We would like to find A, ξ so that $R(z) = (P_0(z) : P_0(z)(A-z)^{-1}\xi)$.

Writing

$$\xi_0 = \xi, \quad \xi_1 = A\xi, \quad \ldots, \quad \xi_{d-1} = A^{d-1}\xi, \tag{21}$$

the requirements for A, ξ become (see (20)) that

$$P_0(A)\xi = 0 \tag{22}$$

holds along with

$$\begin{aligned} T_0(z)\xi_{d-1} + T_1(z)\xi_{d-2} + \cdots + T_{d-1}(z)\xi_0 \\ = -[P_1(z)e_1 + P_2(z)e_2 + \cdots + P_d(z)e_d]. \end{aligned} \tag{23}$$

Here e_j denotes the jth unit vector.

Since each T_j has degree j exactly, (23) can be solved for $\xi_0, \ldots, \xi_{d-1}$, and since the right member of (23) by assumption spans $\mathbf{C}^d$ as z runs through $\mathbf{C}$,

$\xi_0, \ldots, \xi_{d-1}$ must actually be a basis for $\mathbf{C}^d$. Then we may simply take A to be the companion matrix of $P_0(z)$ relative to this basis, which by definition means that $P_0(z)$ is the minimal polynomial of A, in particular that (22) holds, and that $A\xi_{j-1} = \xi_j$ $(1 \le j \le d-1)$, i.e. (21), holds.

Thus A, ξ have the required properties, finishing the proof of Theorem 4.2. ∎

So far we have discussed the general relationship between rational embeddings and linear data (A, ξ). Next we turn to the relationship between given polynomial data $Q(z, \bar{z})$ and linear data (A, ξ) connected to the polynomial as in the case of quadrature domains, i.e., as in (5). However, we do not assume that the data really come from a quadrature domain. For simplicity, we treat only the case that $d = n$ in the above notation, i.e., the case that the vector ξ in the linear data is cyclic for A. We shall also consider decompositions of the polynomial $Q(z, \bar{z})$ into sum of squares.

Let

$$Q(z, \bar{z}) = \sum_{j,k=0}^{d} \alpha_{jk} z^j \bar{z}^k$$

be any self-adjoint polynomial, normalized so that $\alpha_{dd} = 1$, and set also

$$P(z) = \sum_{j=0}^{d} \alpha_j z^j.$$

Then

$$|P(z)|^2 - Q(z, \bar{z}) = \sum_{j,k=0}^{d} (\alpha_{jd}\alpha_{dk} - \alpha_{jk}) z^j \bar{z}^k. \tag{24}$$

THEOREM 4.3: *The following conditions are equivalent:*

(a) *The matrix* $A(\alpha) = (\alpha_{jd}\alpha_{dk} - \alpha_{jk})_{j,k=0}^{d-1}$ *is positive definite.*

(b) *There exists a linear transformation* A *of* $\mathbf{C}^d$ *with a cyclic vector* ξ *so that* $P(A) = 0$ *and*

$$\frac{Q(z, \bar{z})}{|P(z)|^2} = 1 - \|(A - z)^{-1}\xi\|^2. \tag{25}$$

(c) *There exist polynomials* $Q_k(z)$ *of degree* k *(exactly),* $0 \le k < d$*, with the property*

$$Q(z, \bar{z}) = |P(z)|^2 - \sum_{k=0}^{d-1} |Q_k(z)|^2. \tag{26}$$

In (c), *the* Q_k *are uniquely determined if the leading coefficients are required to be positive.*

Proof: (a)⇒(b). Assume that $A(\alpha)$ is positive definite. It is well-known that every positive definite matrix is the Gram matrix of some basis. Thus there exist linearly independent vectors $v_k \in \mathbf{C}^d$, $0 \le k < d$, satisfying

$$\langle v_j, v_k \rangle = \alpha_{jd}\alpha_{dk} - \alpha_{jk}.$$

By this, relation (24) becomes

$$|P(z)|^2 - Q(z,\bar{z}) = \|V(z)\|^2, \tag{27}$$

where $V(z) = \sum_{j=0}^{d-1} v_j z^j$, a vector-valued polynomial.

It follows that R: $\mathbf{P}_1 \to \mathbf{P}_d$, defined in terms of homogeneous coordinates in $\mathbf{P}_d$ by $R(z) = (P(z) : V(z))$, is a rational map of degree d such that the image of R spans $\mathbf{P}_d$. Thus Theorem 4.2 provides linear data (A, ξ) with ξ cyclic such that

$$V(z) = P(z)(A - z)^{-1}\xi.$$

Now assertion (b) follows.

(b)⇒(c). To achieve the decomposition (26) we use again the polynomials (18) and note that (19) still holds in the present context. We then only have to orthonormalize the vectors $\xi, A\xi, \ldots, A^{d-1}\xi$:

$$e_0 = \frac{\xi}{\|\xi\|},$$
$$e_1 = \frac{A\xi - \langle A\xi, e_0\rangle e_0}{\|\cdots\|},$$
$$e_2 = \frac{A^2\xi - \langle A^2\xi, e_1\rangle e_1 - \langle A^2\xi, e_0\rangle e_0}{\|\cdots\|},$$

etc. We get

$$\xi = \|\xi\| e_0 = c_0 e_0 \quad (c_0 > 0),$$
$$A\xi = c_1 e_1 + \langle A\xi, e_0\rangle e_0 \quad (c_1 > 0),$$
$$A^2\xi = c_2 e_2 + \langle A^2\xi, e_1\rangle e_1 + \cdots \quad (c_2 > 0),$$

and so on, and relation (19) gives

$$-P(z)(A - z)^{-1}\xi = T_0(z)A^{d-1}\xi + \cdots + T_{d-1}(z)\xi$$

$$= T_0(z)(c_{d-1}e_{d-1} + \langle A^{d-1}\xi, e_{d-2}\rangle e_{d-2} + \cdots)$$
$$+T_1(z)(c_{d-2}e_{d-2} + \langle A^{d-2}\xi, e_{d-3}\rangle e_{d-3} + \cdots) + \cdots + T_{d-1}(z)c_0e_0 =$$
$$= c_{d-1}T_0(z)e_{d-1} + (c_{d-2}T_1(z) + \langle A^{d-1}\xi, e_{d-2}\rangle T_0(z))e_{d-2} + \cdots$$
$$+(c_0T_{d-1}(z) + \langle A\xi, e_0\rangle T_{d-2}(z) + \cdots)e_0$$
$$= Q_0(z)e_{d-1} + Q_1(z)e_{d-2} + \cdots + Q_{d-1}(z)e_0,$$

where

$$Q_k(z) = c_{d-1-k}T_k(z) + O(z^{k-1}).$$

Hence $Q_k(z)$ is a polynomial of degree k with leading coefficient $c_{d-1-k} > 0$, and (26) now follows by inserting the above expression for $P(z)(A-z)^{-1}\xi$ into (25) and using that the e_j are orthonormal.

(c)⇒(a). If assertion (c) is assumed to be true, then the vector-valued polynomial

$$V(z) = (Q_0(z), Q_1(z), \ldots, Q_{d-1}(z))$$

satisfies (27). Expanding $V(z)$ along increasing powers of z gives $V(z) = \sum_{j=0}^{d-1} v_j z^j$ where the v_j are linearly independent vectors. Then (27) and (24) show that $A(\alpha)$ is the Gram matrix of the v_j. Hence $A(\alpha)$ is positive definite, proving (a).

It remains to prove the uniqueness of the decomposition (26). For this we observe that there exists a simple algorithm of finding the polynomials Q_k. Indeed, first observe that the coefficient of $\overline{z}^d$ in $Q(z,\overline{z})$ is $P(z)$. Hence the polynomial $F_{d-1}(z,\overline{z}) = |P(z)|^2 - Q(z,\overline{z})$ has degree $d-1$ in each variable. By assumption the coefficient γ_1 of $z^{d-1}\overline{z}^{d-1}$ in F_{d-1} is positive, so that

$$F_{d-1}(z,\overline{z}) = {\gamma_1}^{1/2}\overline{z}^{d-1}Q_{d-1}(z) + O(z^{d-1}, \overline{z}^{d-2}).$$

Therefore the polynomial $Q_{d-1}(z)$ is determined by $F_{d-1}(z,\overline{z})$.

Proceeding by descending recurrence in k ($k < d-1$) we are led to the polynomial

$$F_k(z,\overline{z}) = F_{k+1}(z,\overline{z}) - |Q_{k+1}(z)|^2$$

which has as leading term a positive constant γ_k times $z^k\overline{z}^k$. Then necessarily

$$F_k(z,\overline{z}) = {\gamma_k}^{1/2}\overline{z}^k Q_k(z) + O(z^k, \overline{z}^{k-1}).$$

Thus $Q_k(z)$ is determined by $F_k(z,\overline{z})$. And so on until we end by setting $F_0(z,\overline{z}) = \gamma_0 = |Q_0(z,\overline{z})|^2 > 0$.

This finishes the proof of Theorem 4.3. ∎

By sections 1 and 2, all of the above discussions apply to quadrature domains. Notice that the $\overline{z}$ in, e.g., the right member of (5) can be replaced by z by just conjugating the coefficients of U^* and ξ. Alternatively, one may use the U^* and ξ belonging to the conjugate domain $\Omega^* = \{\overline{z} \in \mathbf{C}; z \in \Omega\}$, which is a quadrature domain whenever Ω is. Then (5) is really on the form as it appears in (b) of Theorem 4.3. Thus, in view of (6) we have, for example:

COROLLARY 4.4: *A quadrature domain of order d is rationally isomorphic to the intersection of the rational normal curve of degree d in $\mathbf{P}_d$ and the complement of a real affine ball.*

Also, by Theorem 4.3 we get

COROLLARY 4.5: *The equation for the boundary of a quadrature domain of order d can be written uniquely on the form*

$$\sum_{j=0}^{d-1} |Q_j(z)|^2 = |P(z)|^2,$$

where each Q_j is a polynomial of degree j with positive leading coefficient and where P is the monic polynomial of degree d which vanishes (with the right multiplicities) at the quadrature nodes.

In addition to the statements in the corollary we remark that the leading coefficient of $Q_{d-1}(z)$ is

$$c_0 = \|\xi\| = \left(\frac{\text{Area}(\Omega)}{\pi}\right)^{1/2}.$$

The quadrature data of Ω (i.e., the nodes and the weights) are determined by the knowledge of the rational function (4). By comparing

$$Q(z,\overline{z}) = |P(z)|^2 - |Q_{d-1}(z)|^2 + O(z^{d-2}, \overline{z}^{d-2})$$

with the middle term in (4), and by considering the behaviour at infinity of these functions, we find that the rational function (4) coincides with

$$\frac{c_0 Q_{d-1}(z)}{P(z)}.$$

Therefore the quadrature data of Ω are in a natural bijection with the pair of polynomials $P(z), Q_{d-1}(z)$. The other polynomials $Q_{d-2}(z), \ldots, Q_0(z)$ determine the domain Ω (via its defining function) and they depend on $(d-1)^2$ real parameters; see also [G1], Theorem 10.

In the particular case $d = 2$ we have $\deg(P(z)) = 2$ and

$$Q(z, \bar{z}) = |P(z)|^2 - |az + b|^2 - c,$$

where $a, c > 0$ and $b \in \mathbf{C}$. Examples 6.1 and 6.2 below treat such cases.

As an application of Theorem 4.3 we discuss the structure of the exponential transform:

$$E_\Omega(z, \overline{w}) = \exp\left[\frac{-1}{\pi} \int_\Omega \frac{dA(\zeta)}{(\zeta - z)(\bar{\zeta} - \overline{w})}\right], \quad |z|, |w| \gg 0,$$

of a quadrature domain Ω which possesses rotational symmetries. In the above notation, for large values of $|z|$ we have, by (9),

$$E_\Omega(z, \bar{z}) = \frac{|P(z)|^2 - \sum_{k=0}^{d-1} |Q_k(z)|^2}{|P(z)|^2}.$$

PROPOSITION 4.6: *Let Ω be a quadrature domain and let ϵ be a primitive root of unity, of order n. Assume that $\Omega = \epsilon\Omega$.*

Then for all $z \in \mathbf{C}$, $|P(\epsilon z)| = |P(z)|$ and $|Q_k(\epsilon z)| = |Q_k(z)|$, $0 \leq k \leq d - 1$.

Proof: From the very definition of the exponential transform, it follows by the change of variables $\zeta \mapsto \epsilon^{-1}\zeta$ that $E_\Omega(z, \overline{w}) = E_\Omega(\epsilon z, \epsilon^{-1}\overline{w})$. In virtue of the uniqueness of the decomposition (26) above, the ϵ-rotational symmetry of the functions $|P|$ and $|Q_k|$ follows. ∎

If a monic polynomial Q satisfies $|Q(z)| = |Q(\epsilon z)|$, then $Q(z)$ is a product of a factor z^m, for some $m \geq 0$, and factors like $(z^n - a^n)$, $a \neq 0$. This remark leads to the following result, noted also in [G2].

COROLLARY 4.7: *Let Ω be a quadrature domain of order d and let ϵ be a primitive root of unity, of order d, so that $\Omega = \epsilon\Omega$.*

Then there exists a complex number $a \neq 0$, so that the nodes of Ω are $\epsilon^k a$, $0 \leq k \leq d - 1$, and the defining equation of $\partial\Omega$ has the form

$$|z^d - a^d|^2 = \sum_{k=0}^{d-1} c_k |z^k|^2,$$

where $c_k > 0$, $0 \leq k \leq d - 1$.

Proof: The case $d = 1$ leads to the defining equation $|z - a|^2 = c$, hence Ω is a disk.

Assume that $d > 1$. By degree reasons, it is clear that $|Q_k(z)| = c_k|z^k|^2$, for all $0 \leq k \leq d-1$. By Theorem 4.3, the constants c_k must be positive.

For the leading polynomial $P(z)$, of degree d, there are two possibilities. Either $P(z) = z^d - a^d$, with $a \neq 0$, or $P(z) = z^d$. The latter case is excluded, because Ω would be defined by a polynomial in $|z|^2$, hence it would be a union of annuli, which is not a quadrature domain. ∎

5. Rational embedding of Schwarz reflections into projective space

The main issue in this section is to give a geometric version in $\mathbf{P}_n$ of the Schwarzian reflection (the antianalytic reflection) in the boundary of a quadrature domain. In particular this will give a criterion which complements Theorems 4.2 and 4.3 and ensures that the domain (or open set) defined in terms of general data such as $R(z)$, (A, ξ) or $Q(z, \bar{z})$ is really a quadrature domain.

We start from a situation as in Section 4. Let

$$R\colon \mathbf{P}_1 \to \mathbf{P}_n$$

be a rational embedding of some degree d and normalized so that $R(\infty) = 0$ in terms of given embeddings $\mathbf{C} \subset \mathbf{P}_1$, $\mathbf{C}^n \subset \mathbf{P}_1$. Let k denote the dimension of the linear span of the image $R(\mathbf{P}_1)$. It is easy to see that necessarily $k \leq d$. In case $k = d$ we know by Theorem 4.2 that R is of the form $R(z) = (A - z)^{-1}\xi$ for some $A\colon \mathbf{C}^n \to \mathbf{C}^n$, $0 \neq \xi \in \mathbf{C}^n$, but for the moment we do not assume that. In analogy with the case of quadrature domains we define the open set:

$$\Omega = \{z \in \mathbf{C}; \|R(z)\| > 1\}. \tag{28}$$

We first remark that the singular points a in the boundary of Ω are given by the equation $\langle R'(a), R(a)\rangle = 0$. Here we know that $R'(a) \neq 0$ since R is an embedding. On the other hand, the Hessian $H(a)$ at a of the defining equation $\|R(z)\|^2 = 1$ is

$$H(a) = \begin{pmatrix} \langle R'(a), R'(a)\rangle & \langle R''(a), R(a)\rangle \\ \langle R(a), R''(a)\rangle & \langle R'(a), R'(a)\rangle \end{pmatrix}.$$

In particular $\operatorname{rank} H(a) \geq 1$, which shows that a is either an isolated point or a singular double point of $\partial\Omega$. In the case of a non-isolated singular point a on the boundary of a quadrature domain Ω it is known that a is a cusp or a double tangency point, and in this case $\operatorname{rank} H(a) = 1$; see [G1].

Now let

$$C = R(\mathbf{P}_1) \subset \mathbf{P}_n$$

be the image curve of R and let $\mathbf{B}_n = \{z \in \mathbf{C}^n; \|z\| < 1\}$ denote the affine unit ball, which we also consider as a subset of $\mathbf{P}_n$. Set also

$$C_+ = C \setminus \overline{\mathbf{B}_n}, \quad C_- = C \cap \mathbf{B}_n.$$

Thus, by (28), $\Omega = R^{-1}(C_+) = R^{-1}(\mathbf{P}_n \setminus \overline{\mathbf{B}_n})$.

That the map R, or curve C, has degree d means that C meets any generic hyperplane $L \subset \mathbf{P}_n$ in exactly d points. Given any point $a \in C$ we may take this hyperplane to be

$$L = L_a = \{z \in \mathbf{P}_n; \langle z, a\rangle = 1\}.$$

(The scalar product is written in the fixed affine chart $z = (1 : z)$.) Then we obtain a multivalued (1 to d) reflection map $J\colon C \to C$, namely defined by

$$J\colon a \mapsto L_a \cap C, \quad a \in C.$$

If $a \in C \cap \partial\mathbf{B}_n$, then $a \in L_a \cap C$. Thus $C \cap \partial\mathbf{B}_n$ consists of fixed points for one branch of J. Locally it is possible to select single-valued branches of J, and these are antianalytic since one of the factors in the inner product defining L_a is conjugated.

From the inequality

$$1 = |\langle z, a\rangle| \le \|z\|\|a\| \tag{29}$$

for $z \in L_a$ it follows that $J(a) \subset C_+$ if $a \in C_-$. Moreover, by considering the equality case we find that $J(a) \setminus \{a\} \subset C_+$ if $a \in C \cap \partial\mathbf{B}_n$.

Pulling J back to $\mathbf{P}_1$ via R gives a multivalued reflection map $\overline{S} : \mathbf{P}_1 \to \mathbf{P}_1$ defined by $R \circ \overline{S} = J \circ R$ or, equivalently,

$$\langle R(z), R(\overline{S}(z))\rangle = 1 \quad (z \in \mathbf{P}_1). \tag{30}$$

Its single-valued branches $\overline{S}_1(z), \ldots, \overline{S}_d(z)$ are antianalytic, and from the previous considerations we get

PROPOSITION 5.1: *The multivalued reflection $z \mapsto (\overline{S}_j(z))_{j=1}{}^d$ satisfies, for an appropriate numbering of the branches:*

(a) *If $z \in \mathbf{P}_1 \setminus \overline{\Omega}$ then $\overline{S}_j(z) \in \Omega$ for $1 \le j \le d$.*
(b) *If $z \in \partial\Omega$ then $\overline{S}_1(z) = z$ and $\overline{S}_j(z) \in \Omega$ for $2 \le j \le d$.*

From (b) of the proposition it is clear that the branch of $\overline{S}$ called $\overline{S}_1$ is identical with the conjugate of the Schwarz function $S(z)$ of $\partial\Omega$; see (2). (A local Schwarz function, satisfying (2), exists whenever $\partial\Omega$ is analytic, something which is certainly satisfied in the present context when $\partial\Omega = R^{-1}(\partial\mathbf{B}_n)$.) Recall now the characterization [AS], [D] of (bounded) quadrature domains, saying that these are exactly those domains Ω for which there is a Schwarz function of $\partial\Omega$ which is meromorphic in all Ω. From this we obtain the following criterion for the domain (or open set) (28) to be a quadrature domain.

THEOREM 5.2: *Let Ω be defined by (28) in terms of a rational embedding R: $\mathbf{P}_1 \to \mathbf{P}_n$ satisfying $R(\infty) = 0$. Then the following conditions are equivalent:*

(a) *Ω is a quadrature domain, or a finite disjoint union of such domains.*
(b) *There exists a single-valued antianalytic selection in all Ω of the branch $\overline{S}_1$ of $\overline{S}$.*
(c) *There exists a single-valued antianalytic selection J_1: $C_+ \to C$ of that branch of J: $C \to C$ which has $C \cap \partial\mathbf{B}_n$ as fixed points.*

Remark: The assumption $R(\infty) = 0$ in the theorem is almost superfluous. The only significance of it is that it together with definition (28) prevents that $\infty \in \overline{\Omega}$, i.e., that Ω is unbounded. For unbounded quadrature domains, in particular those which are dense in $\mathbf{C}$, some new phenomena appear which will not be discussed in this paper. We refer to [Sa] for a complete analysis and classification of unbounded quadrature domains.

Certainly, Theorem 5.2 is merely a translation of a by now classical characterization ([AS], [D]) of quadrature domains. The main point for us is that the new condition (c) is a purely geometric statement whereas the classical characterization (essentially (b)) is stated in terms of function theoretic quantities.

There are also other ways to inject quadrature domains into projective spaces so that the Schwarzian reflection becomes a geometric object. Perhaps the most natural way is via the injection $z \mapsto (z, S(z))$ of Ω into the completion in $\mathbf{P}_2$ of the algebraic curve $\Gamma = \{(z,w) \in \mathbf{C}^2; Q(z,w) = 0\}$. On Γ the Schwarzian reflection simply becomes $(z,w) \mapsto (\overline{w}, \overline{z})$. A slight drawback with this realization is that in most cases Ω is not really embedded in $\mathbf{P}_2$ because Γ almost always has singular points (self-intersection etc.). See [G2] for some details concerning this. On the other hand, the reflection map is single-valued.

Returning now to our embedding R, a natural question is: if Ω in (28) happens to be a quadrature domain, must then the given embedding R agree with the conjugate R_Ω of the "canonical" embedding $z \to (U^* - \overline{z})^{-1}\xi$ given in terms of some linear data (U, ξ) of Ω? Since the linear data are determined only up to unitary equivalence we cannot expect R and R_Ω to really agree. However, the following theorem shows that under some (necessary) degree restrictions, R and R_Ω are as similar as they can possibly be.

THEOREM 5.3: *Let R: $\mathbf{P}_1 \longrightarrow \mathbf{P}_n$ be a rational embedding of degree k such that $R(\infty) = 0$ and such that $\Omega = R^{-1}(\mathbf{P}_n \setminus \overline{\mathbf{B}_n})$ is a quadrature domain of order d. Then $k \geq d$. If $k = d$ then, with R_Ω as above, there exists an isometric embedding i: $\mathbf{P}_d \longrightarrow \mathbf{P}_n$ with the property that $R = i \circ R_\Omega$.*

Remark: The conclusion may also be stated along the lines of Theorem 4.2, namely as saying that R necessarily is of the form $R(z) = (A - z)^{-1}\xi$ for some linear transformation A of $\mathbf{C}^n$ and some non-zero vector ξ in $\mathbf{C}^n$. The assumption that Ω is a quadrature domain of order equal to k then replaces the assumption in Theorem 4.2 that the image of R should span a space of dimension k.

Proof of Theorem 5.3: The Schwarzian reflection for a quadrature domain of order d has exactly d branches when extended as a multivalued map to all $\mathbf{P}_1$. Thus the reflection maps $\overline{S}$ and J in Theorem 5.2 must have at least d branches if the equivalent conditions are satisfied, hence the degree k of R must be at least d, proving the first assertion of the theorem.

Next assume $k = d$ and write

$$R(z) = (P_0(z) : P_1(z) : \dots : P_n(z))$$

where $\deg P_0 = d > \deg P_j$, $1 \leq j \leq n$, and P_0 is normalized to be monic. Let $Q(z, \overline{z})$ be the normalized polynomial (1) of $\partial\Omega$ and let $S(z)$ be the Schwarz function of $\partial\Omega$. Since necessarily $R(\partial\Omega) \subset \partial\mathbf{B}_n$ we have

$$P_0(z)\overline{P_0(\overline{S(z)})} - \sum_{j=1}^{n} P_j(z)\overline{P_j(\overline{S(z)})} = 0,$$

for $z \in \partial\Omega$, and therefore identically.

On the other hand, $Q(z, S(z)) = 0$ and Q is the minimal polynomial of the algebraic function $S(z)$. In view of normalizations and degree assumptions it follows that the two polynomials occurring above are identical:

$$P_0(z)\overline{P_0(w)} - \sum_{j=1}^{n} P_j(z)\overline{P_j(w)} = Q(z, \overline{w})$$

for all $z, w \in \mathbf{C}$.

Such a relation also holds for the "canonical" embedding R_Ω, which we write as $R_\Omega(z) = (Q_0(z) : Q_1(z) : \dots : Q_d(z))$ ($\deg Q_0 = d > \deg Q_j$, $1 \leq j \leq d$, Q_0

monic). Thus

$$P_0(z)\overline{P_0(w)} - \sum_{j=1}^{n} P_j(z)\overline{P_j(w)} = Q_0(z)\overline{Q_0(w)} - \sum_{j=1}^{d} Q_j(z)\overline{Q_j(w)}$$

identically. It follows that $P_0(z) = Q_0(z)$ (since these are the coefficients of $\overline{w}^d$) and that R and R_Ω are isometrically related:

$$\langle R(z), R(w)\rangle = \langle R_\Omega(z), R_\Omega(w)\rangle \quad (z, w \in \mathbf{C}).$$

Since the range of R_Ω spans the vector space $\mathbf{C}^d$, there is a uniquely defined isometry $V\colon \mathbf{C}^d \longrightarrow \mathbf{C}^n$, such that $V(R_\Omega(z)) = R(z), z \in \mathbf{C}$. At the level of projective spaces V induces a linear embedding $i\colon \mathbf{P}_d \longrightarrow \mathbf{P}_n$ and $R = i \circ R_\Omega$, as desired. This finishes the proof of Theorem 5.3. The remark after the theorem follows immediately from Theorem 4.2, since by the above isometry the range of R spans a space of dimension d. ∎

The following example shows that the case $k > d$ in the theorem really may occur, and that the conclusion need not hold then. Take the embedding $R\colon \mathbf{P}_1 \longrightarrow \mathbf{P}_2$ given in the standard charts of coordinates by the formula

$$R(z) = \left(\frac{1}{2^{1/2}z}, \frac{1}{2^{1/2}z^2}\right).$$

Then the degree of R is two and $R^{-1}(\mathbf{P}_2 \setminus \mathbf{B}_2)$ is the unit disk, a quadrature domain of order one.

We next specialize to simply connected quadrature domains. It is well-known [AS] that the (bounded) simply connected quadrature domains are exactly the conformal images of the unit disc $\mathbf{B}_1$ under rational functions, univalent in $\mathbf{B}_1$ and with the poles off $\overline{\mathbf{B}_1}$. The order of the quadrature domain equals the degree of the rational function, i.e., the number of times it attains (in $\mathbf{P}_1$) almost every value (see for instance [AS]).

Thus let $\Omega = \phi(\mathbf{B}_1)$ with ϕ rational as above. Let d be the order of Ω as a quadrature domain and let $R\colon \mathbf{P}_1 \to \mathbf{P}_n$ be a rational embedding of degree d such that $R(\infty) = 0$ and so that Ω and R are related by (28), and set $C = R(\mathbf{P}_1)$. Then the composed map

$$r = R \circ \phi\colon \mathbf{P}_1 \to \mathbf{P}_n$$

has degree d^2 and maps $\mathbf{B}_1$ via Ω onto $C_+ = C \setminus \overline{\mathbf{B}_n}$. Clearly, r maps every pole of ϕ to $0 \in \mathbf{P}_n$.

As we saw above, when the multivalued reflection J in C was pulled back via R to $\mathbf{P}_1$ we got the multivalued reflection $\overline{S}$ in $\mathbf{P}_1$, of which the (conjugate of the)

Schwarz function is one branch. If we pull this reflection back one step further, via ϕ, we get the usual single-valued reflection map $\zeta \mapsto 1/\overline{\zeta}$ in $\partial\mathbf{B}_1$. In fact, by (30) we have for the Schwarz function $S(z)$ that $\langle R(z), R(\overline{S(z)})\rangle = 1$, $z \in \overline{\Omega}$. Since $\overline{S(\phi(\zeta))} = \phi(1/\overline{\zeta})$, for $|\zeta| = 1$ and therefore identically, the rational map $r(\zeta) = R(\phi(\zeta))$ satisfies the duality relation:

$$\langle r(\zeta), r(1/\overline{\zeta})\rangle = 1, \tag{31}$$

identically on $\mathbf{P}_1$, proving the claim.

Thus, in some extended sense, the rational map r is commuting with the reflections in the boundaries of the unit balls of $\mathbf{C}$ and respectively $\mathbf{C}^d$.

Finally, taking degrees into account we can prove an analogue of Theorem 5.3 for r:

COROLLARY 5.4: *Let r: $\mathbf{P}_1 \longrightarrow \mathbf{P}_n$ be a rational map, set $C = r(\mathbf{P}_1)$ and assume that $0 \in C$ and that the restriction of r to $\mathbf{B}_1$ is an isomorphism of $\mathbf{B}_1$ onto C_+.*

Then C_+ is rationally isomorphic to a simply connected quadrature domain $\Omega \subset \mathbf{C}$. Assume that $\deg(r) \le d^2$, where d denotes the order of Ω. Then

$$r = i \circ R_\Omega \circ \phi,$$

with R_Ω as in Theorem 5.3, i: $\mathbf{P}_d \to \mathbf{P}_n$ an isometric embedding and ϕ: $\mathbf{P}_1 \longrightarrow \mathbf{P}_1$ a rational map of degree d, which is one to one on $\mathbf{B}_1$.

In particular the degree of r is d^2.

Proof: Let R: $\mathbf{P}_1 \longrightarrow \mathbf{P}_n$ be a rational embedding which parametrizes, by Lüroth's theorem, the curve C. We may take it to satisfy $R(\infty) = 0$. Let d' be the degree of R, or, which is the same, the degree of C.

We can define the antianalytic map J: $C_+ \longrightarrow C$ by the formula

$$J(r(\zeta)) = r(1/\overline{\zeta}) \quad (\zeta \in \mathbf{B}_1).$$

Since necessarily $r(\partial\mathbf{B}_1) \subset \partial C_+ \subset \partial\mathbf{B}_n$ we obtain

$$\langle r(1/\overline{\zeta}), r(\zeta)\rangle = 1 \quad (\zeta \in \partial\mathbf{B}_1).$$

Hence, by analytic continuation, the same identity holds for $\zeta \in \mathbf{B}_1$. Therefore

$$J(r(\zeta)) \in L_{r(\zeta)} \cap C \quad (\zeta \in \mathbf{B}_1).$$

We also have $J(r(\zeta)) = r(\zeta)$ for all $\zeta \in \partial\mathbf{B}_1$ and by assumption C_+ is isomorphic to the unit disk, hence it is simply connected.

Set $\Omega = R^{-1}(C_+)$. Thus Ω is a simply connected quadrature domain of some order d, and necessarily $d \le d'$ because the defining equation $\|R(z)\| > 1$ of Ω has degree $2d'$ and may not be irreducible. We remark next that the map $\phi = R^{-1} \circ r$ is well defined, hence analytic, hence rational. By construction ϕ is a conformal transformation of the unit disk $\mathbf{B}_1$ onto Ω. Since the degree of the rational map ϕ equals the order d of Ω we get

$$d^2 \ge \deg(r) = \deg(R)\deg(\phi) = dd' \ge d^2.$$

Therefore $d = d'$ and Theorem 5.3 can be applied to the rational embedding R. This proves the conclusion of Corollary 5.4. ∎

6. Examples

The complexity of computations of the basic objects attached to a quadrature domain increases very fast with the order. At least for order two quadrature domains such computations are possible, and they have appeared, from different perspectives, in [AS], [D], [G1], [Sa]. Below we show how the matrix U and the vector ξ enter into the picture of order two quadrature domains.

6.1. THE LIMAÇON. Let $z = w^2 + bw$, where $|w| < 1$ and $b \ge 2$. Then z describes a quadrature domain Ω of order 2, whose boundary has the equation

$$Q(z,\overline{z}) = |z|^4 - (2+b^2)|z|^2 - b^2 z - b^2\overline{z} + 1 - b^2 = 0;$$

see for instance [DL], Section 5.1.

The Schwarz function of Ω has a double pole at $z = 0$, whence the 2×2-matrix U is nilpotent. Moreover, we know that

$$|z|^4 \|(U^* - \overline{z})^{-1}\xi\|^2 = |z|^4 - P(z,\overline{z}).$$

Therefore

$$\|(U^* + \overline{z})\xi\|^2 = (2+b^2)|z|^2 + b^2 z + b^2\overline{z} + b^2 - 1,$$

or equivalently, $\|\xi\|^2 = 2 + b^2$, $\langle U^*\xi, \xi\rangle = b^2$ and $\|U^*\xi\|^2 = b^2 - 1$.

Consequently the linear data of the quadrature domain Ω are

$$U^* = \begin{pmatrix} 0 & \frac{b^2-1}{(b^2-2)^{1/2}} \\ 0 & 0 \end{pmatrix}, \quad \xi = \begin{pmatrix} \frac{b^2}{(b^2-1)^{1/2}} \\ (\frac{b^2-2}{b^2-1})^{1/2} \end{pmatrix}.$$

This shows in particular that the pair (U, ξ) is subject to some other restrictions than $U^2 = 0$ and ξ being a cyclic vector for U^*. For an abstract version of these restrictions, see [P1].

The rational embedding of the conjugated domain Ω^* can easily be computed from the definition (20):

$$R(1:z) = \Big(-z^2 : \frac{b^2}{(b^2-1)^{1/2}} z + \frac{b^2-1}{(b^2-1)^{1/2}} : \Big(\frac{b^2-2}{b^2-1}\Big)^{1/2} z \Big).$$

Notice that in this situation $\Omega^* = \Omega$, therefore the rational conformal map ϕ: $\mathbf{D} \longrightarrow \Omega^*$ is $z = \phi(w) = w^2 + bw$. According to the previous computations, the rational map $r(w) = R(1 : w^2 + bw)$ satisfies the symmetry condition (31).

6.2. TWO DISTINCT NODES. (a) Suppose that Ω is a quadrature domain with the quadrature distribution

$$u(f) = af(0) + bf(1),$$

where we choose the constants a and b to be positive numbers. Then $P(z) = z(z-1)$ and

$$\overline{z}(\overline{z}-1)(U^* - \overline{z})^{-1}\xi = -U^*\xi + \xi - \overline{z}\xi.$$

Therefore the equation of the boundary of Ω is

$$Q(z,\overline{z}) = |z(z-1)|^2 - \|U^*\xi - \xi + \overline{z}\xi\|^2.$$

According to the quadrature relations (4) we obtain

$$\|\xi\|^2 = \frac{a+b}{\pi}, \qquad \langle U\xi, \xi\rangle = \frac{b}{\pi}.$$

Let us denote $\|U^*\xi\|^2 = c$. Then the defining polynomial becomes

$$Q(z,\overline{z}) = |z(z-1)|^2 - \pi^{-1}(a|z-1|^2 + b(|z|^2 - 1)) - c.$$

The constant c actually depends on a and b via, for instance, the relation Area$(\Omega) = a + b$, or, whenever $a = b$, the fact that $Q(1/2, 1/2) = 0$; see [G1], Corollary 10.1.

We can choose an orthonormal basis with respect to which we have

$$U^* = \begin{pmatrix} 0 & \alpha \\ 0 & 1 \end{pmatrix}, \quad \xi = \begin{pmatrix} \beta \\ \gamma \end{pmatrix}.$$

The matricial elements α, β, γ are then subject to the relations

$$|\beta|^2 + |\gamma|^2 = \pi^{-1}(a+b), \quad \overline{\alpha}\beta\overline{\gamma} + |\gamma|^2 = \pi^{-1}b, \qquad |\alpha|^2|\gamma|^2 + |\gamma|^2 = c.$$

An inspection of the arguments shows that the above system of equations has real solutions α, β, γ given by the formulas

$$\alpha^2 = \frac{(\pi c - b)^2}{\pi(a+b)c - b^2}, \quad \beta^2 = \frac{a^{-2}}{\pi(a-b) + \pi^2 c}, \quad \gamma^2 = \frac{\pi(a+b)c - b^2}{\pi(a-b)+\pi^2 c}.$$

Let us remark that, if $a = b > \pi/4$, the constant c is effectively computable, as mentioned earlier, and becomes

$$c = \frac{1}{16} + \frac{a}{2\pi}.$$

This again illustrates the special nature of the pair (U, ξ). A simple computation shows that the corresponding canonical embedding of the domain $\Omega = \Omega^*$ is

$$R(1:z) = (z(z-1) : \beta(1-z) - \alpha\gamma : \gamma z).$$

We remark that in both of the above examples, the matrix U and the vector ξ are uniquely determined, as soon as we require that U is upper triangular.

(b) In complete analogy, we can treat the case of two nodes with equal weights as follows.

Assume that the nodes are fixed at ± 1. Hence $P(z) = z^2 - 1$. The defining equation of the quadrature domain Ω of order two with these nodes is

$$Q(z, \overline{z}) = (|z+1|^2 - r^2)(|z-1|^2 - r^2) - c,$$

where r is a positive constant and $c \geq 0$ is chosen so that either Ω is a union of two disjoint open disks (in which case $c = 0$), or $Q(0,0) = 0$. For details see [G2]. A short computation yields

$$Q(z, \overline{z}) = z^2\overline{z}^2 - 2rz\overline{z} - z^2 - \overline{z}^2 + \alpha(r),$$

where

$$\alpha(r) = \begin{cases} (1-r^2)^2, & r < 1, \\ 0, & r \geq 1 \end{cases}$$

Equivalently, for the derivation of the formula of Q we can invoke Corollary 4.7, which gives for $\partial\Omega$ the equation

$$|z^2 - 1|^2 = c_1|z|^2 + c_0,$$

with positive constants c_k, $k = 0, 1$. Then we proceed as above.

Exactly as in the preceding two situations, the identification

$$|P(z)|^2(1 - \|(U^* - \overline{z})^{-1}\xi\|^2) = Q(z, \overline{z}) \tag{32}$$

leads to (for example) the following simple linear data:

$$\xi = \begin{pmatrix} \sqrt{2}r \\ 0 \end{pmatrix}, \quad U^* = \begin{pmatrix} 0 & \frac{\sqrt{2}r}{\sqrt{1-\alpha(r)}} \\ \frac{\sqrt{1-\alpha(r)}}{\sqrt{2}r} & 0 \end{pmatrix}.$$

We leave to the reader the verification of formula (32).

6.3. Domains corresponding to a nilpotent matrix. To give a basic example for the class of domains discussed in Section 4 but which is not a quadrature domain, we consider the nilpotent matrix A and the cyclic vector ξ:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \xi = \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

where a, b, c are complex numbers, $c \neq 0$. A simple computation shows that

$$\|(A-z)^{-1}\xi\|^2 = \left|\frac{a}{z} + \frac{b}{z^2} + \frac{c}{z^3}\right|^2 + \left|\frac{b}{z} + \frac{c}{z^2}\right|^2 + \left|\frac{c}{z}\right|^2.$$

Therefore the equation of the associated domain is

$$|z|^6 < |az^2 + bz + c|^2 + |bz^2 + cz|^2 + |cz^2|^2.$$

According to Proposition 5.1, the reflection in the boundary of this domain maps the exterior completely into its interior.

The rational embedding associated to this example is

$$R(1:z) = (-z^3 : az^2 + bz + c : bz^2 + cz : cz^2).$$

Similarly, one can compute without difficulty the corresponding objects associated to a nilpotent Jordan block and an arbitrary cyclic vector of it. For instance, the nilpotent $n \times n$-Jordan block and the vector $\xi = (0, 0, \ldots, 0, -1)$ give precisely the Veronese embedding:

$$R(1:z) = (z^n : 1 : z : \ldots : z^{n-2} : z^{n-1}).$$

Compare the remarks preceding Theorem 4.2.

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